Relative randomness and real closed fields

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Abstract

We show that for any real number, the class of real numbers less random than it, in the sense of rK-reducibility, forms a countable real closed subfield of the real ordered field. This generalizes the well-known fact that the computable reals form a real closed field.

With the same technique we show that the class of differences of computably enumerable reals (d.c.e. reals) and the class of computably approximable reals (c.a. reals) form real closed fields. The d.c.e. result was also proved nearly simultaneously and independently by Ng [6].

Lastly, we show that the class of d.c.e. reals is properly contained in the class or reals less random than Ω (the halting probability), which in turn is properly contained in the class of c.a. reals, and that neither the first nor last class is a randomness class (as captured by rK-reducibility).

Key words: relative randomness, real closed field, rK-reducibility, d.c.e. real, c.a. real

1 Introduction

What does it mean for one real number to be less random than another? In attempts to answer this question, computability theorists have invented a variety of preorders (reflexive and transitive relations) on (various representations of) the reals, almost all of which are motivated by and involve the following idea.

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Roughly, we say a real number x is random if the initial segments of the binary representation of its fractional part are patternless (incompressible). More precisely, we say x is *random* iff

$$\exists c\, \forall n\ [K(\widetilde{x}\restriction n) \geq n - c]\,,$$

where $\widetilde{x}$ is the binary sequence of the binary representation of the fractional part of x and $K(\sigma)$ is the prefix-free program size (Kolmogorov) complexity of the binary string σ. Roughly, we say a real number x is less random than a real number y if the initial segments of the binary representation of the fractional part of x are less patternless (more compressible) than those of y. More precisely, we say x is less random than y, iff $x \leq_r y$ for some measure of relative randomness $\leq_r$. A preorder on the reals (or binary sequences) $\leq_r$ is a *measure of relative randomness* iff for all $x, y \in \mathbb{R}$

$$x \leq_r y \Rightarrow \exists c\, \forall n\ [K(\widetilde{x}\restriction n) \leq K(\widetilde{y}\restriction n) + c]\,.$$

(I really should write 'less or *equally* random as' instead of 'less random than' throughout, but I prefer the latter, less precise phrase for its brevity.)

In this paper we focus on one particularly nice measure of relative randomness, relative Kolmogorov (rK) reducibility, introduced by Downey, Hirschfeldt, and LaForte [2], and explore some consequences of grouping real numbers by this measure.

Before beginning, let us set some conventions and notation. We will represent real numbers in binary (without cofinitely many ones), using strings for integer parts and (infinite) sequences for fractional parts. However, when doing computations with reals behind the scenes we will use approximating rationals (as quotients of integers).

Definition 1.1

- $\mathbb{N} = \omega = \{0, 1, 2, \ldots\}$.
- For ease of reading, we will use quantifiers somewhat informally, and all unbounded quantification will take place over the natural numbers or objects coded by natural numbers.
- c.p.f. abbreviates 'computable partial function(s)'.
- Let $\{0,1\}^n$ denote the set of binary strings of length n (functions from n to $\{0,1\}$), let $\{0,1\}^{<\omega}$ denote the set of binary strings (partial functions from ω to $\{0,1\}$), and let $\{0,1\}^\omega$ denote the set of (infinte) binary sequences (functions from ω to $\{0,1\}$).
- $\langle\quad\rangle$ will delimit ordered tuples and sequences.
- For $n \in \mathbb{N}$, 0^n denotes the string $\langle\overbrace{00\cdots0}^{n \text{ times}}\rangle$.
- For $x \in \mathbb{R}$, let $\widetilde{x}$ be the binary sequence of the binary expansion of the fractional part of x.

- For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, let $x \upharpoonright n$ be the binary representation of x up to and including the first n bits past the binary point.

Lastly, herein we refer to some standard computational classes of reals. A real number x is *computable* iff there exists a computable sequence of rationals $\langle q_s \rangle_{s\in\mathbb{N}}$ *converging effectively* to x, that is, there is a computable function $e : \mathbb{N} \to \mathbb{N}$ such that for all $N \in \mathbb{N}$

$$s \geq e(N) \to |q_s - x| \leq 2^{-N}.$$

Equivalently, x is computable iff $\widetilde{x}$ is a computable function. A real number x is *computably enumerable (c.e.)* iff there is a computable increasing sequence of rationals converging to x. A real number x is *a difference of c.e. reals (d.c.e.)* iff there exist c.e. reals y, z such that $x = y - z$. Finally, a real number x is *computably approximable (c.a.)* iff there is a computable sequence of rationals converging to x (with no further restrictions on the sequence).

Let $\mathbb{R}_{\text{c}}$ denote the class of computable reals, $\mathbb{R}_{\text{c.e.}}$ the class of c.e. reals, $\mathbb{R}_{\text{d.c.e.}}$ the class of d.c.e. reals, and $\mathbb{R}_{\text{c.a.}}$ the class of c.a. reals. These classes are properly contained in each other: $\mathbb{R}_{\text{c}} \subset \mathbb{R}_{\text{c.e.}} \subset \mathbb{R}_{\text{d.c.e.}} \subset \mathbb{R}_{\text{c.a.}}$ (see [1] by Ambos-Spies, Weihrauch, and Zheng for instance).

2 Real Closed Fields

In this section we show that for any real number, the class of real numbers less random than it, in the sense of rK-reducibility, forms a countable real closed subfield of the real ordered field.

Let us first begin with the definition of rK-reducibility introduced by Downey et al. [2].

Definition 2.1 For α, $\beta \in \{0,1\}^\omega$, $\alpha \leq_{\text{rK}} \beta$ iff

$$\exists \text{c.p.f. } \varphi :\subseteq \{0,1\}^{<\omega} \times \mathbb{N} \to \{0,1\}^{<\omega}\ \exists k\ \forall n\ \exists i < k\ [\varphi(\beta \upharpoonright n, i)\downarrow = \alpha \upharpoonright n]\,.$$

In this case we also write $\alpha = [\varphi, k]^\beta$.

For ease of reading we will abuse notation and write $x \leq_{\text{rK}} y$ for real numbers x and y when we really mean

$$\exists \text{c.p.f. } \varphi :\subseteq \{0,1\}^{<\omega} \times \mathbb{N} \to (\{0,1\}^{<\omega})^2\ \exists k\ \forall n\ \exists i < k\ [\varphi(\widetilde{y} \upharpoonright n, i)\downarrow = \langle \sigma, \widetilde{x} \upharpoonright n \rangle]\,,$$

where $\sigma \in \{0,1\}^{<\omega}$ is the binary representation of the integer part of x (with, say, the first bit coding whether x is positive or negative). Again, in this case we write $x = [\varphi, k]^y$.

Choosing from the many measures of relative randomness, we focus on rK-reducibility because of its nice properties.

Theorem 2.2 (Downey et al. [2])

☞ $\leq_{\mathrm{rK}}$ *is a preorder (reflexive and transitive relation), and so, working with equivalence classes, gives rise to a degree structure.*

☞ *If* $\alpha \leq_{\mathrm{rK}} \beta$*, then* $\exists c\, \forall n\ [K(\alpha\restriction n) \leq K(\beta\restriction n) + c]$*. So* $\leq_{\mathrm{rK}}$ *is indeed a measure of relative randomness. It relates to prefix-free program size complexity.*

☞ *If* $\alpha \leq_{\mathrm{rK}} \beta$*, then* $\exists c\, \forall n\ [K(\alpha\restriction n|\beta\restriction n) \leq c]$*, where* $K(\sigma|\tau)$ *is the prefix-free program size complexity of* σ *given (using oracle)* τ*. In this sense* $\leq_{\mathrm{rK}}$ *is an* exact *measure of relative randomness.*

☞ *If* $\alpha \leq_{\mathrm{rK}} \beta$*, then* $\alpha \leq_{\mathrm{T}} \beta$*. So rK also relates to computational complexity.*

Using rK-reducibility, we can group reals into randomness classes. For the rest of this section fix some $y \in \mathbb{R}$ and $\mathbb{R}_y := \{x \in \mathbb{R} : x \leq_{\mathrm{rK}} y\}$, the class of reals less random than y. Perhaps surprisingly, (every) $\mathbb{R}_y$ has tame algebraic/analytic structure. It is a real closed field.

This generalizes the well-known fact that $\mathbb{R}_{\mathrm{c}}$, the class of computable reals, forms a real closed field (see [7] by Pour-El and Richards for instance) in the following sense. $x \in \mathbb{R}$ is computable iff $\widetilde{x} \leq_{\mathrm{T}} \widetilde{0}$ (the sequence of all zeros) iff $x \leq_{\mathrm{rK}} 0$ (remember that rK-reducibility implies T-reducibility) implying that $x \leq_{\mathrm{rK}} y$. Thus $\mathbb{R}_{\mathrm{c}} = \mathbb{R}_0 \subseteq \mathbb{R}_y$, that is, the class of computable reals is the randomness class $\mathbb{R}_0$ (or $\mathbb{R}_a$, for any computable real a), which is contained in the (arbitrary) randomness class $\mathbb{R}_y$.

As a first step to showing $\mathbb{R}_y$ is a real closed field, we introduce a large class of functions under which $\mathbb{R}_y$ is closed, the weakly computable locally Lipschitz functions.

Definition 2.3 Let $s \in \mathbb{N}^+$, $E \subseteq \mathbb{R}^s$ be open, and $f : E \to \mathbb{R}$.

- f is *locally Lipschitz* iff for each $x \in E$ there is an open set $E_0 \subseteq E$ containing x such that

$$\exists M \in \mathbb{R}^+\, \forall \boldsymbol{x}, \boldsymbol{y} \in E_0\ [|f(\boldsymbol{x}) - f(\boldsymbol{y})| \leq M|\boldsymbol{x} - \boldsymbol{y}|]\,,$$

  where $|\quad|$ is the Euclidean norm.

- f is *weakly computable* iff $f\restriction E \cap \mathbb{Q}^s$ uniformly outputs computable reals in the following sense:

$$\exists \text{c.p.f. } \hat{f} :\subseteq \mathbb{Q}^s \times \mathbb{N} \to \mathbb{Q}\, \forall \boldsymbol{q}\, \forall n \left[\boldsymbol{q} \in E \cap \mathbb{Q}^s \to \hat{f}(\boldsymbol{q}, n)\downarrow = f(\boldsymbol{q})\restriction n\right]$$

- f is *weakly computable locally Lipschitz (w.c.l.L.)* iff f is weakly computable and locally Lipschitz.

Remark 2.4 It is easy to see that weakly computable Lipschitz functions are computable, and computable functions are weakly computable (for a definition of 'computable' in this sense see [7] by Pour-El and Richards, for instance). Also, as a fact from elementary real analysis, locally Lipschitz functions on compact domains are Lipschitz. Thus w.c.l.L. functions on compact domains are computable functions. We could use the stronger notion of ‘computable function’ instead of ‘weakly computable function’ throughout, but weak computability suffices, and its critereion is slightly easier to check.

The following two lemmas and short comment thereafter explain why w.c.l.L. functions interact so well with rK-reducibility.

Lemma 2.5 *If $f : E \subseteq \mathbb{R}^s \to \mathbb{R}$ is locally Lipschitz, then for all $\boldsymbol{x} \in E$*

$$\exists C\, \forall n > C \ \left[|f(\boldsymbol{x}) - f(\boldsymbol{x} \upharpoonright n)| < 2^{C-n} \right],$$

where $\boldsymbol{x} \upharpoonright n = \langle x_0 \upharpoonright n, \ldots, x_{s-1} \upharpoonright n \rangle$.

Lemma 2.6 *Let $x, y \in \mathbb{R}$ and $C, n \in \mathbb{N}$ with $n > C$. If $|x - y| < 2^{C-n}$, then there exist $a < 2$ and $\rho \in \{0,1\}^C + 1$ such that $[x + (-1)^a 0.0^{n-C-1} {}^\frown \rho] \upharpoonright n = y \upharpoonright n$.*

Using 2.5 and 2.6 we can now show that $\mathbb{R}_y$ is closed under w.c.l.L. functions. The basic idea is this. Suppose $\boldsymbol{x} \in (\mathbb{R}_y)^s$ and f is a weakly computable locally Lipschitz function. Since f is locally Lipschitz, the first n bits of $f(\boldsymbol{x})$, which we want via an rK-computation from y, are just the first n bits of $[f(\boldsymbol{x} \upharpoonright n) +$ fuzz$]$, which we can get via an rK-computation from y since the fuzz is of bounded variability. The hypothesis of weak computability on f ensures that the partial function we build witnessing rK-reducibility is computable.

Lemma 2.7 *Let $s \in \mathbb{N}^+$. If $\boldsymbol{x} \in (\mathbb{R}_y)^s$, $f : E \subseteq \mathbb{R}^s \to \mathbb{R}$ is w.c.l.L, and $\boldsymbol{x} \in E$, then $f(\boldsymbol{x}) \in \mathbb{R}_y$.*

Of course, this result is vacuous unless w.c.l.L. functions actually exist. They certainly do. To see this, let us dig up a helpful fact from real analysis: if f is differentiable on E (with E open), then f is locally Lipschitz on E. Since $+, -, \cdot : \mathbb{R}^2 \to \mathbb{R}$, $/ : \mathbb{R} \times \mathbb{R} \setminus \{0\} \to \mathbb{R}$, and $\sqrt{\ } : \mathbb{R}^+ \to \mathbb{R}$ are differentiable and certainly weakly computable, they are examples of w.c.l.L. functions. Key examples, in fact, because with these and just a little more real analysis we can show

Theorem 2.8 *$\langle \mathbb{R}_y, +, \cdot, < \rangle$ is a countable real closed subfield of the real ordered field.*

With our preparation, the proof is not too difficult. First we need to show that $\mathbb{R}_y$ forms a countable ordered subfield of the real ordered field. This follows from rK-reducibility implying Turing reducibility (for the countability part) and 2.7 since subtraction and division are w.c.l.L. functions (for the ordered subfield part).

Last, we need to show that the field is real closed. Given a positive real from $\mathbb{R}_y$, its square root is also in $\mathbb{R}_y$ by 2.7, since square root is a w.c.l.L. function. Also given an odd degree polynomial with coefficients in $\mathbb{R}_y$, we need to show it has a root in $\mathbb{R}_y$. The polynomial certainly has a root in $\mathbb{R}$. Applying the Implicit Function Theorem to our polynomial with its coefficients replaced with variables, we can show that the root finding function is w.c.l.L. It is locally Lipschitz since the Implicit Function Theorem promises it is differentiable, and it is weakly computable since after substituting rational coefficients into our polynomial we can use a binary search algorithm (just like in the proof of $\mathbb{R}_c$ forming a real closed field) to find the root.

3 The Reals Less Random Than Ω

We now narrow our view and look more closely at one particular randomness class, the class of reals less random than the halting probability Ω. Downey et al. [2] show that, in analogy to every c.e. set being T-reducible to the halting set, every c.e. real is rK-reducible to Ω; in symbols, $\mathbb{R}_{\text{c.e.}} \subseteq \mathbb{R}_\Omega$. In fact, even more is true.

Proposition 3.1 $\mathbb{R}_{\text{d.c.e.}} \subseteq \mathbb{R}_\Omega \subseteq \mathbb{R}_{\text{c.a.}}$.

The first inclusion holds since $\mathbb{R}_\Omega$ is closed under subtraction, and the second holds since rK-reducibility implies T-reducibility, Ω is T-equivalent to $\emptyset'$, and, by a result of Ho [3], every $\emptyset'$-computable real is c.a.

Moreover,

Theorem 3.2 *$\mathbb{R}_{\text{d.c.e.}}$ and $\mathbb{R}_{\text{c.a.}}$ form countable real closed fields.*

$\mathbb{R}_{\text{d.c.e.}}$ and $\mathbb{R}_{\text{c.a.}}$ are clearly countable since there are only countably many computable sequences of rationals. They are also real closed fields via the same proof used in 2.8, because they are closed under w.c.l.L. functions. This closure follows from the lemmas belows.

Lemma 3.3 (Ambos-Spies et al. [1]) *$x \in \mathbb{R}_{\text{d.c.e.}}$ iff there is a computable sequence of rationals $\langle q_i \rangle_{i\in\mathbb{N}}$ converging to x such that $\sum_{i\in\mathbb{N}} |q_{i+1} - q_i| < \infty$.*

Recall that a sequence of reals $\langle x_i \rangle_{i\in\mathbb{N}}$ is *computable* iff there is a double computable sequence of rationals $\langle q_{ij} \rangle_{i,j\in\mathbb{N}}$ and a computable function $e : \mathbb{N}^2 \to \mathbb{N}$ such that for all i, n

$$j \geq e(i,n) \to |q_{ij} - x_i| \leq 2^{-n}.$$

Lemma 3.4 (Ambos-Spies et al. [1]) *If a computable sequence of reals $\langle x_i \rangle_{i\in\mathbb{N}}$ converges to x such that $\sum_{i\in\mathbb{N}} |x_{i+1} - x_i| < \infty$, then $x \in \mathbb{R}_{\text{d.c.e.}}$.*

Lemma 3.5 *Let $s \in \mathbb{N}^+$. If $\boldsymbol{x} \in (\mathbb{R}_{\text{d.c.e.}})^s$, $f : E \subseteq \mathbb{R}^s \to \mathbb{R}$ is w.c.l.L, and $\boldsymbol{x} \in E$, then $f(\boldsymbol{x}) \in \mathbb{R}_{\text{d.c.e.}}$.*

Lemma 3.6 (Zheng and Weihrauch [8]) *If a computable sequence of reals $\langle x_i \rangle_{i\in\mathbb{N}}$ converges to x, then $x \in \mathbb{R}_{\text{c.a.}}$.*

Lemma 3.7 *Let $s \in \mathbb{N}^+$. If $\boldsymbol{x} \in (\mathbb{R}_{\text{c.a.}})^s$, $f : E \subseteq \mathbb{R}^s \to \mathbb{R}$ is w.c.l.L, and $\boldsymbol{x} \in E$, then $f(\boldsymbol{x}) \in \mathbb{R}_{\text{c.a.}}$.*

That $\mathbb{R}_{\text{d.c.e.}}$ forms a real closed field was also proved nearly simultaneously and independently by Ng [6].

4 Proper Containment

So $\mathbb{R}_{\text{d.c.e.}} \subseteq \mathbb{R}_\Omega \subseteq \mathbb{R}_{\text{c.a.}}$, and all three classes form countable real closed fields. Is $\mathbb{R}_{\text{d.c.e.}} = \mathbb{R}_\Omega$ or $\mathbb{R}_\Omega = \mathbb{R}_{\text{c.a.}}$? (Note that both can not be true since $\mathbb{R}_{\text{d.c.e.}} \subset \mathbb{R}_{\text{c.a.}}$.) An affirmative answer for either case would yield intriguing alternate characterizations of both classes involved. However

Theorem 4.1 $\mathbb{R}_{\text{d.c.e.}} \neq \mathbb{R}_\Omega$.

Theorem 4.2 $\mathbb{R}_\Omega \neq \mathbb{R}_{\text{c.a.}}$.

The proof of 4.1 is a finite injury priority argument, in which we construct $\alpha \in \{0,1\}^\omega$ such that $\alpha \leq_{\text{rK}} \Omega$ and $0.\alpha$ is not a d.c.e. real. Instead of making $\alpha \leq_{\text{rK}} \Omega$ directly, we construct a c.e. real $0.\beta$ such that $\alpha \leq_{\text{rK}} \beta$; here we use the fact that all c.e. reals are rK-reducible to Ω. The construction is a priority argument written in the style of Lempp's notes [4], where we meet, for all pairs of c.e. reals $\langle x, y\rangle$, the following requirements.

$$\mathcal{R}_{x,y} : 0.\alpha \neq x - y \wedge \exists\theta\ \alpha = [\theta, 2]^\beta.$$

To ensure $0.\alpha \neq x - y$, we flip a big bit of α exponentially often so that, eventually, $x - y$ will tire and fail to keep up. To ensure $\alpha = [\theta, 2]^\beta$ we put big gaps of zeros in β and redefine θ by changing β in its gaps whenever α changes on certain big bits. The gaps are big enough so that, in the end, $0.\beta$ will be a c.e. real.

The proof of 4.2 is also a finite injury priority argument, in which we construct $\alpha \in \{0,1\}^\omega$ meeting for all pairs of a c.p.f. and a natural number $\langle \varphi, k\rangle$, the following requirements.

$$\mathcal{R}_{\varphi,k} : \alpha \neq [\varphi, k]^\Omega$$

Meeting each requirement is done simply by picking a section of bits of length $k+1$ near the end of α and flipping it through $k+1$ different incarnations. Eventually, $[\varphi, k]^\Omega$ will fail to keep up, at which point we stop flipping. Since we construct α in a computable fashion, we get that $0.\alpha \in \mathbb{R}_{\text{c.a.}}$.

Let us end with one last question. We now know that $\mathbb{R}_{\text{d.c.e.}} \subset \mathbb{R}_\Omega \subset \mathbb{R}_{\text{c.a.}}$. Is $\mathbb{R}_{\text{d.c.e.}}$ or $\mathbb{R}_{\text{c.a.}}$ a randomness class (as captured by rK-reducibility)? That is, does $\mathbb{R}_{\text{d.c.e.}}$ or $\mathbb{R}_{\text{c.a.}}$ equal $\mathbb{R}_y$ for *any* real number y?

By the proper inclusion of 4.1 and the technique in the proof of 4.2, it follows that, here again, the answer is negative.

Theorem 4.3 *For all $y \in \mathbb{R}$, $\mathbb{R}_{\text{d.c.e.}} \neq \mathbb{R}_y$ and $\mathbb{R}_{\text{c.a.}} \neq \mathbb{R}_y$.*

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