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RELATIVE RANDOMNESS AND REAL CLOSED FIELDS

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Abstract. We show that for any real number, the class of real numbers less random than it, in the sense of rK-reducibility, forms a countable real closed subfield of the real ordered field. This generalizes the well-known fact that the computable reals form a real closed field.

With the same technique we show that the class of differences of computably enumerable reals (d.c.e. reals) and the class of computably approximable reals (c.a. reals) form real closed fields. The d.c.e. result was also proved nearly simultaneously and independently by Ng (Keng Meng Ng, Master's Thesis, National University of Singapore, in preparation).

Lastly, we show that the class of d.c.e. reals is properly contained in the class of reals less random than Ω (the halting probability), which in turn is properly contained in the class of c.a. reals, and that neither the first nor last class is a randomness class (as captured by rK-reducibility).

§1. Introduction. What does it mean for one real number to be less random than another? In attempts to answer this question, to measure the relative randomness of reals, computability theorists have invented a variety of preorders (reflexive and transitive relations) on (various representations of) the reals, almost all of which are motivated by the prefix-free complexity characterization of absolute randomness. Roughly, we say an infinite binary sequence α, thought of as the binary representation of the fractional part of a real number, is random if its initial segments are incompressible (patternless). More precisely, we say α is **random** iff

$$(\exists c)(\forall n)\ K(\alpha \upharpoonright n) \geq n - c,$$

where $K(\sigma)$ is the prefix-free complexity of the binary string σ.

Recently, Downey, Hirschfeldt, and LaForte [2] introduced a new preorder called relative Kolmogorov (rK) reducibility. Based on conditional prefix-free complexity, rK-reducibility turns out to be both a natural, general measure of relative randomness *and* a measure of relative computability (as a refinement of Turing reducibility), making it a promising tool in the study of the interplay between algoritmic randomness and traditional computability theory.

In this paper we explore one aspect of this interplay by grouping reals into randomness classes (as captured by rK-redcubility) and seeing how these relate

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to several well-studied computational classes of reals. We also investigate the algebraic and analytic properties of both types of classes.

Before beginning, let us set some conventions and notation. $\mathbb{N}$ and ω denote the set of natural numbers $\{0, 1, 2, \dots\}$. For ease of reading, we use quantifiers somewhat informally, and all unbounded quantification takes place over the natural numbers or objects coded by natural numbers. C.p.f. abbreviates 'computable partial function(s)'. ${}^n2$ denotes the set of binary strings of length n (functions from n to 2), ${}^{<\mathbb{N}}2$ denotes the set of binary strings (functions from initial segments of ω to 2), and ${}^{\mathbb{N}}2$ denotes the set of (infinte) binary sequences (functions from ω to 2). $|\sigma|$ denotes the length of a binary string σ. For $n \in \mathbb{N}$, 0^n denotes the string of n zeros. For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, $x \restriction n$ denotes the truncation of the binary expansion of x (both the integer and fractional part) up to and including the first n bits past the binary point. $\langle \quad \rangle$ delimits ordered tuples and sequences, and for each $s \in \mathbb{N}^+$, let $\backslash \quad \backslash : \mathbb{N}^s \to \mathbb{N}$ be a lexicographically strictly increasing computable bijection (coding function).

Let us also recall the following standard computational classes of reals. A real number x is **computable** iff there exists a computable sequence of rationals $\langle q : s \in \mathbb{N}\rangle$ converging effectively to x, that is, there is a computable function $e : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$

$$s \geq e(n) \to |q_s - x| \leq 2^{-n}.$$

Equivalently, x is computable iff the binary sequence of the binary exapansion of the fractional part of x is a computable function. A real number x is **computably enumerable (c.e.)** iff there is a computable increasing sequence of rationals converging to x. A real number x is **a difference of c.e. reals (d.c.e.)** iff there exist c.e. reals y, z such that $x = y - z$. A real number x is **computably approximable (c.a.)** iff there is a computable sequence of rationals converging to x (with no further restrictions on the sequence). Let $\mathbb{R}_{\text{c}}$ denote the class of computable reals, $\mathbb{R}_{\text{c.e.}}$ the class of c.e. reals, $\mathbb{R}_{\text{d.c.e.}}$ the class of d.c.e. reals, and $\mathbb{R}_{\text{c.a.}}$ the class of c.a. reals. These classes are properly contained in each other: $\mathbb{R}_{\text{c}} \subset \mathbb{R}_{\text{c.e.}} \subset \mathbb{R}_{\text{d.c.e.}} \subset \mathbb{R}_{\text{c.a.}}$ (see [1] by Ambos-Spies, Weihrauch, and Zheng for instance).

§2. Real Closed Fields. We begin with the definition of rK-reducibility.

Definition 2.1 (Downey et al. [2]). For $\alpha, \beta \in {}^{\mathbb{N}}2$, $\alpha \leq_{\text{rK}} \beta$ iff

$$(\exists c)(\forall n)\ K(\alpha \restriction n | \beta \restriction n) \leq c,$$

where $K(\sigma|\tau)$ is the conditional prefix-free complexity of σ given τ as input (see [5] for more details).

It is straightforward to check that $\leq_{\text{rK}}$ is indeed a preorder on ${}^{\mathbb{N}}2$ and that the following useful equivalencies hold.

Theorem 2.2 (Downey et al. [2]). For $\alpha, \beta \in {}^{\mathbb{N}}2$,

$$\begin{aligned} & \alpha \leq_{\text{rK}} \beta \\ \Leftrightarrow\ & (\exists c)(\forall n)\ C(\alpha \restriction n | \beta \restriction n) \leq c \\ \Leftrightarrow\ & (\exists \text{c.p.f. } \varphi :\subseteq {}^{<\mathbb{N}}2 \times \mathbb{N} \to {}^{<\mathbb{N}}2)(\exists c)(\forall n)(\exists i < c)\ \varphi(\beta \restriction n, i)\downarrow = \alpha \restriction n, \end{aligned}$$

where $C(\sigma|\tau)$ is the conditional (plain) complexity of σ given τ as input (again, see [5] for more details).

Also, as alluded to in the introduction, we have the following nice properties of $\leq_{\mathrm{rK}}$.

THEOREM 2.3 (Downey et al. [2]). For $\alpha, \beta \in {}^{\mathbb{N}}2$,

- $\alpha \leq_{\mathrm{rK}} \beta \Rightarrow (\exists c)(\forall n)\ K(\alpha\restriction n) \leq K(\beta\restriction n) + c$;
- $\alpha \leq_{\mathrm{rK}} \beta \Rightarrow (\exists c)(\forall n)\ C(\alpha\restriction n) \leq C(\beta\restriction n) + c$;
- $\alpha \leq_{\mathrm{rK}} \beta \Rightarrow \alpha \leq_{\mathrm{T}} \beta$.

Thinking of 'rK-reducible to' to mean 'less random than' (though 'less than or equally random as' would be more precise), we can group real numbers into randomness classes. To this end, we use the last characterization of $\leq_{\mathrm{rK}}$ in Theorem 2.2 rephrased slightly in terms of reals. For $x, y \in \mathbb{R}$, $x \leq_{\mathrm{rK}} y$ iff

$$(\exists \text{c.p.f. } \varphi :\subseteq \mathbb{Q} \times \mathbb{N} \to \mathbb{Q})(\exists c)(\forall n)(\exists i < c)\ \varphi(y\restriction n, i)\downarrow = x\restriction n.$$

In this case we write $x = [\varphi, c]^y$. Now, given $y \in \mathbb{R}$, let

$$\mathbb{R}_y = \{x \in \mathbb{R} : x \leq_{\mathrm{rK}} y\},$$

the class of reals less random than y.

Perhaps surprisingly, each $\mathbb{R}_y$ has tame algebraic and analytic structure; each is a real closed field. This generalizes the well-known fact that $\mathbb{R}_c$, the class of computable reals, forms a real closed field [1] in the following sense. $x \in \mathbb{R}$ is computable iff $x \leq_{\mathrm{T}} \emptyset$ (identifying x with the binary sequence of the binary expansion of its fractional part) iff $x \leq_{\mathrm{rK}} 0$ (remember that rK-reducibility is a refinement of T-reducibility). Thus $\mathbb{R}_c = \mathbb{R}_0$, that is, the class of computable reals is the randomness class $\mathbb{R}_0$ (or $\mathbb{R}_a$, for any computable real a). Notice also that $\mathbb{R}_0 \subseteq \mathbb{R}_y$ for all y.

For the rest of this section, fix a randomness class $\mathbb{R}_y$. As a first step to showing $\mathbb{R}_y$ is a real closed field, we introduce a large class of functions under which $\mathbb{R}_y$ is closed, the weakly computable locally Lipschitz functions.

DEFINITION 2.4. Let $s \in \mathbb{N}^+$, $E \subseteq \mathbb{R}^s$ be open, and $f : E \to \mathbb{R}$.

- f is **locally Lipschitz** iff for each $x \in E$ there is an open set $E_0 \subseteq E$ containing x on which f is Lipschitz, that is

  $$(\exists M \in \mathbb{R}^+)(\forall \vec{x}, \vec{y} \in E_0)\ |f(\vec{x}) - f(\vec{y})| \leq M|\vec{x} - \vec{y}|,$$

  where $|\ \ |$ is the Euclidean norm.
- f is **weakly computable** iff $f\restriction E \cap \mathbb{Q}^s$ uniformly outputs computable reals in the following sense:

  $$(\exists \text{c.p.f. } \hat{f} :\subseteq \mathbb{Q}^s \times \mathbb{N} \to \mathbb{Q})(\forall \vec{q})(\forall n)\ \vec{q} \in E \cap \mathbb{Q}^s \to \hat{f}(\vec{q}, n)\downarrow = f(\vec{q})\restriction n$$

- f is **weakly computable locally Lipschitz (w.c.l.L.)** iff f is weakly computable and locally Lipschitz.

[1] see [8] by Pour-El and Richards for instance.

Remark 2.5. It is easy to see that weakly computable Lipschitz functions are computable, and computable functions are weakly computable. [2] Also, as a fact from elementary real analysis, locally Lipschitz functions on compact domains are Lipschitz. Thus w.c.l.L. functions on compact domains are computable functions. We could use the stronger notion of 'computable function' instead of 'weakly computable function' throughout, but weak computability suffices, and its critereion is slightly easier to check.

The following two lemmas and short comment thereafter explain why w.c.l.L. functions interact so well with rK-reducibility.

Lemma 2.6. If $f : E \subseteq \mathbb{R}^s \to \mathbb{R}$ is locally Lipschitz, then for all $\vec{x} \in E$

$$(\exists C)(\forall n > C)\ |f(\vec{x}) - f(\vec{x} \restriction n)| < 2^{C-n},$$

where $\vec{x} \restriction n = \langle x_0 \restriction n, \ldots, x_{s-1} \restriction n \rangle$.

Proof. Suppose $f : E \subseteq \mathbb{R}^s \to \mathbb{R}$ is locally Lipschitz and $\vec{x} \in E$. Then there is an open $E_0 \subseteq E$ containing $\vec{x}$ such that f is Lipschitz on E_0. Thus

$$\begin{aligned} & (\exists M \in \mathbb{Q}^+)(\forall \vec{y} \in E_0)\ |f(\vec{x}) - f(\vec{y})| \leq M|\vec{x} - \vec{y}| \\ \Rightarrow\ & (\exists M \in \mathbb{Q}^+)(\forall^\infty n)\ |f(\vec{x}) - f(\vec{x} \restriction n)| \leq M\sqrt{s}2^{-n} \\ & (\text{since } (\forall^\infty n)\ \vec{x} \restriction n \in E_0 \text{ and } (\forall^\infty n)\ |\vec{x} - \vec{x} \restriction n| \leq \sqrt{s2^{-2n}} = \sqrt{s}2^{-n}) \\ \Rightarrow\ & (\exists C)(\forall n > C)\ |f(\vec{x}) - f(\vec{x} \restriction n)| < 2^{C-n}. \end{aligned}$$

⊣

Lemma 2.7. Let $x, y \in \mathbb{R}$ and $C, n \in \mathbb{N}$ with $n > C$. If $|x - y| < 2^{C-n}$, then there exist $j < 2$ and $\rho \in {}^{C+1}2$ such that $[y + (-1)^j 0.0^{n-C-1} {}^\frown \rho] \restriction n = x \restriction n$.

Proof. An easy exercise in binary addition. ⊣

Using Lemma 2.6 and Lemma 2.7 we can now show that $\mathbb{R}_y$ is closed under w.c.l.L. functions. The basic idea is this. Suppose $\vec{x} \in (\mathbb{R}_y)^s$ and f is a weakly computable locally Lipschitz function. Since f is locally Lipschitz, the first n bits of $f(\vec{x})$, which we want via an rK-computation from y, are just the first n bits of $[f(\vec{x} \restriction n) +$ fuzz$]$, which we can get via an rK-computation from y since the fuzz is of bounded variability. The hypothesis of weak computability on f ensures that the partial function we build witnessing rK-reducibility is computable.

Lemma 2.8. Let $s \in \mathbb{N}^+$. If $\vec{x} \in (\mathbb{R}_y)^s$, $f : E \subseteq \mathbb{R}^s \to \mathbb{R}$ is w.c.l.L, and $\vec{x} \in E$, then $f(\vec{x}) \in \mathbb{R}_y$.

Proof. For notational niceness let us prove the special case $s = 2$. The general proof is no more difficult. Suppose $\vec{x} = \langle x_0, x_1 \rangle \in (\mathbb{R}_y)^2$, say $x_0 = [\varphi_0, c_0]^y$ and $x_1 = [\varphi_1, c_1]^y$. Since f is locally Lipschitz, there exists, by Lemma 2.6, $C \in \mathbb{N}$ such that for all $n > C$,

$$|f(\vec{x}) - f(\vec{x} \restriction n)| < 2^{C-n}.$$

So by Lemma 2.7, $(\forall n > C)(\exists j < 2)(\exists \rho \in {}^{C+1}2)$

$$\left[f(\vec{x} \restriction n) + (-1)^j 0.0^{n-C-1} {}^\frown \rho\right] \restriction n = f(\vec{x}) \restriction n.$$

[2] For a definition of 'computable' in this sense see [8] by Pour-El and Richards for instance.

Now, list ${}^{C+1}2$ as $\rho_0, \ldots, \rho_{2^{C+1}-1}$, and let $\theta :\subseteq \mathbb{Q} \times \mathbb{N} \to \mathbb{Q}$ be defined by

$$\theta(\tau, \backslash i_0, i_1, j, k\backslash) = \left[f(\varphi_0(\tau, i_0), \varphi_1(\tau, i_1)) + (-1)^j 0.0^{|\tau|-C-1} {}^\frown \rho_k\right] \restriction |\tau|$$

if $i_0 < c_0, i_1 < c_1, j < 2, k < 2^{C+1}, |\tau| > C$ and undefined otherwise. θ is a c.p.f. since f is weakly computable, and for all $n > C$ there is $\backslash i_0, i_1, j, k\backslash < \backslash c_0, c_1, 2, 2^{C+1}\backslash$ such that

$$\begin{aligned} \theta(y \restriction n, \backslash i_0, i_1, j, k\backslash) &= [f(\vec{x} \restriction n) + \text{ correct fuzz}] \restriction n \\ &\quad (\text{since } \vec{x} \leq_{\mathrm{rK}} y) \\ &= f(\vec{x}) \restriction n \end{aligned}$$

So $f(\vec{x}) \leq_{\mathrm{rK}} y$ via a slightly altered constant that only depends on c_0, c_1, and C and a slightly altered c.p.f. θ' that deal with the (finitely many) exceptional $n \leq C$. ⊣

Of course, this result is vacuous unless w.c.l.L. functions actually exist. They certainly do. To see this, let us dig up a helpful fact from real analysis: if f is differentiable on E (with E open), then f is locally Lipschitz on E. Since $+$, $-$, $\cdot$, $/$, and $\sqrt{\ }$ are differentiable and certainly weakly computable, they are examples of w.c.l.L. functions (restricting domains where necessary). Key examples, in fact, because with these and just a little more real analysis we can reach our goal.

Theorem 2.9. $\langle \mathbb{R}_y, +, \cdot, < \rangle$ is a countable real closed field.

Proof. First we show that $\mathbb{R}_y$ forms a countable ordered field. $\mathbb{R}_y$ is nonempty since it contains the computable reals. It is countable since rK-reducibility implies Turing reducibility and the Turing cone below a function is countable. $\mathbb{R}_y$ is certainly ordered by $<$, since it is a set of real numbers. Also, given $a, b \in \mathbb{R}_y$, $a - b$ and a/b (for $b \neq 0$) are both in $\mathbb{R}_y$ by Lemma 2.8, since, as mentioned previously, subtraction and division are w.c.l.L. functions.

Lastly, we show that the field is real closed, that is, every positive real number in $\mathbb{R}_y$ has a square root in $\mathbb{R}_y$, and every odd degree polynomial with coefficients in $\mathbb{R}_y$ has a root in $\mathbb{R}_y$.

A positive real less random then y has a square root less random than y by Lemma 2.8 since $\sqrt{\ }$ (away from 0) is w.c.l.L.

Odd-degree polynomial roots present a little more difficulty. Let $f(x) = c_0 + c_1 x + \cdots + c_m x^m \in \mathbb{R}_y[x]$ be of odd degree. Then f has a root $r \in \mathbb{R}$ and there exists an open interval with rational endpoints (a, b) on which f changes sign and has no other roots. We show $r \in \mathbb{R}_y$.

First off, we may assume without loss of generality that r is a root of multiplicity 1. To see this, note that if r has multiplicity $k > 1$, then k must be odd, because $f(x) = (x - r)^k g(x)$ for some polynomial $g(x)$ which does not change sign on (a, b). (If $g(x)$ changed sign on (a, b), then, by the Intermediate Value Theorem, g and hence f would have a root different from r on (a, b), a contradiction). Thus $f^{(k-1)}$, the $(k-1)$st derivative of f, is an odd-degree polynomial with coefficients in $\mathbb{R}_y$ having r as a root of multiplicity 1, and so we can work with $f^{(k-1)}$ instead of f.

Now, to do the heavy lifting we bring in some more real analysis. Let $O \subseteq \mathbb{R}^{m+1}$ be an open ball containing $\vec{c} = \langle c_0, \dots, c_m \rangle$ and let $F : (a, b) \times O \to \mathbb{R}$ be the polynomial defined by $F(x, \vec{v}) = w_0 + w_1 x + \cdots + w_m x^m$. Then F is continuously differentiable on $(a, b) \times O$, $F(r, \vec{c}) = 0$, and $\partial F/\partial x(r, \vec{c}) = f'(r) \neq 0$ (since r has multiplicity 1). Thus, by the Implicit Function Theorem and its proof (see [9] by Rudin for instance), there are open balls U and V such that

(i) $r \in U \subseteq (a, b)$, $\vec{c} \in V \subseteq O$ (and U has rational endpoints)
(ii) for all $\vec{v} \in V$, $F(x, \vec{v})$ is 1-1 on U
(iii) there is a unique continuously differentiable $G : V \to U$ such that $(\forall \vec{v} \in V)\ F(G(\vec{v}), \vec{v}) = 0$.

With this we show that G is w.c.l.L. and conclude that $r = G(\vec{c}) \in \mathbb{R}_y$ (by Lemma 2.8 since $\vec{c} \in (\mathbb{R}_y)^{m+1}$). Since G is differentiable (by iii), G is locally Lipschitz. So we just need to show that G is weakly computable. For any $\vec{q} \in V \cap \mathbb{Q}^{m+1}$, $F(x, \vec{q})$ is 1-1 on U (by ii) and has exactly one root in U (by iii). So, by applying the standard binary search algorithm on $U \subseteq (a, b)$ (which is independent of $\vec{q}$; see [8] by Pour-El and Richards), $F(x, \vec{q})$ has a computable real root. (Note that for any rational d, $F(d, \vec{q})$ is rational, hence it can be decided whether $F(d, \vec{q}) = 0$, $F(d, \vec{q}) < 0$, or $F(d, \vec{q}) > 0$.) Since $G(\vec{q})$ is that real (by iii), it follows that G is weakly computable. ⊣

§3. The Reals Less Random Than Ω. We now narrow our view and look more closely at one particular randomness class, the class of reals less random than the halting probability Ω. Downey et al. [2] showed that, in analogy to every c.e. set being T-reducible to the halting set, every c.e. real is rK-reducible to Ω; in symbols, $\mathbb{R}_{\text{c.e.}} \subseteq \mathbb{R}_\Omega$. In fact, even more is true.

Proposition 3.1. $\mathbb{R}_{\text{d.c.e.}} \subseteq \mathbb{R}_\Omega \subseteq \mathbb{R}_{\text{c.a.}}$.

Proof. Since $\mathbb{R}_{\text{c.e.}} \subseteq \mathbb{R}_\Omega$ and $\mathbb{R}_\Omega$ is closed under subtraction (by Lemma 2.8), $\mathbb{R}_{\text{d.c.e.}} \subseteq \mathbb{R}_\Omega$. Also, if $x \in \mathbb{R}_\Omega$, then $x \leq_{\text{rK}} \Omega$, implying that $x \leq_{\text{T}} \Omega \equiv_{\text{T}} K$. Therefore the fractional part of x is the characteristic function of a Δ^0_2 set, so that $x \in \mathbb{R}_{\text{c.a.}}$. Thus $\mathbb{R}_\Omega \subseteq \mathbb{R}_{\text{c.a.}}$. ⊣

The last implication in the proof above follows from a result essentially due to Ho [3]:

Lemma 3.2. $x \in \mathbb{R}_{\text{c.a.}}$ iff x is $\emptyset'$-computable (there is a $\emptyset'$-computable sequence of rationals converging effectively to x) iff the fractional part of x is the characteristic function of a Δ^0_2 set.

Moreover, using the same technique from the previous section, we get the following.

Theorem 3.3. $\langle \mathbb{R}_{\text{d.c.e.}}, +, \cdot, < \rangle$ is a countable real closed field.

Theorem 3.4. $\langle \mathbb{R}_{\text{c.a.}}, +, \cdot, < \rangle$ is a countable real closed field.

$\mathbb{R}_{\text{d.c.e.}}$ and $\mathbb{R}_{\text{c.a.}}$ are clearly countable since there are only countably many computable sequences of rationals. They are also real closed fields via the same proof used in Theorem 2.9, because they are closed under w.c.l.L. functions. This closure follows from the lemmas below.

LEMMA 3.5 (Ambos-Spies et al. [1]). $x \in \mathbb{R}_{\text{d.c.e.}}$ iff there is a computable sequence of rationals $\langle q : i \in \mathbb{N} \rangle$ converging to x such that $\sum_{i \in \mathbb{N}} |q_{i+1} - q_i| < \infty$.

Recall that a sequence of reals $\langle x : i \in \mathbb{N} \rangle$ is **computable** iff there is a double computable sequence of rationals $\langle q_{ij} \rangle_{i,j \in \mathbb{N}}$ and a computable function $e : \mathbb{N}^2 \to \mathbb{N}$ such that for all i, n

$$j \geq e(i,n) \to |q_{ij} - x_i| \leq 2^{-n}.$$

LEMMA 3.6 (Ambos-Spies et al. [1]). If a computable sequence of reals $\langle x : i \in \mathbb{N} \rangle$ converges to x such that $\sum_{i \in \mathbb{N}} |x_{i+1} - x_i| < \infty$, then $x \in \mathbb{R}_{\text{d.c.e.}}$.

LEMMA 3.7. Let $s \in \mathbb{N}^+$. If $\vec{x} \in (\mathbb{R}_{\text{d.c.e.}})^s$, $f : E \subseteq \mathbb{R}^s \to \mathbb{R}$ is w.c.l.L, and $\vec{x} \in E$, then $f(\vec{x}) \in \mathbb{R}_{\text{d.c.e.}}$.

PROOF. Let $\vec{x}$ and f be as above. By Lemma 3.5 there is a computable sequence of vectors $\langle \vec{q} : i \in \mathbb{N} \rangle$ from $\mathbb{Q}^s$ that converges to $\vec{x}$ such that $\sum_{i \in \mathbb{N}} |\vec{q}_{i+1} - \vec{q}_i| < \infty$. Since f is locally Lipschitz, there is an open neighborhood E_0 of $\vec{x}$ on which f is Lipschitz, that is

$$(\exists M \in \mathbb{R}^+)(\forall \vec{u}, \vec{v} \in E_0)\ |f(\vec{u}) - f(\vec{v})| \leq M|\vec{u} - \vec{v}| \qquad (\star).$$

Without loss of generality, assume that $\langle \vec{q} : i \in \mathbb{N} \rangle \subseteq E_0$. Since f is weakly computable, $(\forall i)(\forall n)\ |\hat{f}(\vec{q}_i, n) - f(\vec{q}_i)| \leq 2^{-n}$, so that $\langle f(\vec{q}_i) \rangle_{i \in \mathbb{N}}$ is a computable sequence of reals. Also, $\lim_{i \to \infty} f(\vec{q}_i) = f(\lim_{i \to \infty} \vec{q}_i) = f(\vec{x})$ (since locally Lipschitz functions are continuous). Lastly, by $(\star)$,

$$\sum |f(\vec{q}_{i+1}) - f(\vec{q}_i)| \leq M \sum |\vec{q}_{i+1} - \vec{q}_i| < \infty.$$

So $f(\vec{x}) \in \mathbb{R}_{\text{d.c.e.}}$ by Lemma 3.6. ⊣

That $\mathbb{R}_{\text{d.c.e.}}$ forms a real closed field was also proved nearly simultaneously and independently by Ng [7].

LEMMA 3.8 (Zheng and Weihrauch [10]). If a computable sequence of reals $\langle x : i \in \mathbb{N} \rangle$ converges to x, then $x \in \mathbb{R}_{\text{c.a.}}$.

LEMMA 3.9. Let $s \in \mathbb{N}^+$. If $\vec{x} \in (\mathbb{R}_{\text{c.a.}})^s$, $f : E \subseteq \mathbb{R}^s \to \mathbb{R}$ is w.c.l.L, and $\vec{x} \in E$, then $f(\vec{x}) \in \mathbb{R}_{\text{c.a.}}$.

PROOF. This follows from a simplified version of the proof of Lemma 3.7 and from Lemma 3.8. Of course, this also follows by relativiziing the argument for the real closedness of $\mathbb{R}_{\text{c}}$ using Lemma 3.2, but the first approach illustrates the power of w.c.l.L. functions. ⊣

§4. Proper Containment. So $\mathbb{R}_{\text{d.c.e.}} \subseteq \mathbb{R}_\Omega \subseteq \mathbb{R}_{\text{c.a.}}$, and all three classes form countable real closed fields. Is $\mathbb{R}_\Omega$ equal to either $\mathbb{R}_{\text{d.c.e.}}$ or $\mathbb{R}_{\text{c.a.}}$? Notice that both can not be true since $\mathbb{R}_{\text{d.c.e.}} \subset \mathbb{R}_{\text{c.a.}}$. An affirmative answer would yield intriguing alternate characterizations of both classes involved. However, this is not the case.

THEOREM 4.1. $\mathbb{R}_{\text{d.c.e.}} \neq \mathbb{R}_\Omega$.

THEOREM 4.2. $\mathbb{R}_\Omega \neq \mathbb{R}_{\text{c.a.}}$.

Proof of 4.1. We need to construct $\alpha \in {}^{\mathbb{N}}2$ such that $\alpha \leq_{\mathrm{rK}} \Omega$ and $0.\alpha$ is not a d.c.e. real. Instead of making $\alpha \leq_{\mathrm{rK}} \Omega$ directly, we construct a c.e. real $0.\beta$ such that $\alpha \leq_{\mathrm{rK}} \beta$; here we use the fact that all c.e. reals are rK-reducible to Ω. The construction is a $\emptyset'$-priority argument, where we meet, for all c.p.f. $x :\subseteq \mathbb{N} \to \mathbb{Q}$ (possible computable sequences of rationals), the following requirements.

Requirements.

$$\mathcal{R}_x : \qquad (\sum_{s\in\mathbb{N}} |x_s - x_{s+1}| \leq 1 \to 0.\alpha \neq \lim_{s\to\infty} x_s) \wedge (\exists\theta)\ \alpha = [\theta, 2]^{\beta}.$$

These requirements are sufficient since, by a slight modification of Lemma 3.5, every d.c.e. real x has a computable sequence of rationals $\langle x : s \in \mathbb{N}\rangle$ converging to it such that $\sum_{s\in\mathbb{N}} |x_s - x_{s+1}| \leq 1$.

Plan for $\mathcal{R}_x$. To ensure $0.\alpha \neq \lim_{s\to\infty} x_s$, we flip a big bit of α exponentially often so that $0.\alpha$ becomes a super jumping bean. Eventually x_s will tire and fail to keep up, for x_s, being restricted by the condition $\sum_{s\in\mathbb{N}} |x_s - x_{s+1}| \leq 1$, can make at most 2^k jumps of distance at least 2^{-k} (for any k).

More formally, a worker for this requirement proceeds as follows.

1. Pick a big number ('bigbit') n. In particular, n should be bigger than $n_r + \sum_{i\leq n} 2^{2n_i}$, where $n_0, \ldots, n_r$ are all the bigbits mentioned so far in the construction. Extend α and β (which were formerly of lenth n_r) to length n by padding them with zeros. We call $(n_r, n]$ 'n's gap'. Also,

$$\begin{aligned} \theta(\beta \restriction (n+1), 0) &:= \alpha \restriction n ^\frown\langle 0\rangle \\ \theta(\beta \restriction (n+1), 1) &:= \alpha \restriction n ^\frown\langle 1\rangle \\ \theta(\beta \restriction w, 0) &:= \alpha \restriction w \qquad \text{for all } w \in (n_r, n). \end{aligned}$$

2. Wait for

$$\sum_{s=0}^{t} |x_s - x_{s+1}| \leq 1 \quad \text{and} \quad |0.\alpha - x| < \epsilon := 2^{-n-3},$$

(with the convention that all the terms of the sum must be defined) where t is the current stage of the construction. While waiting, each time $\alpha \restriction (n+1)$ changes below n_r (because of higher priority workers), add 2^{-n-1} to $0.\beta$, that is, increment n's gap in β by the minimum amount for each change, and redefine θ for bits $[n_r, n+1)$ just as in (1). Notice that changes in $\alpha \restriction (n+1)$ above n_r require no redefining of θ, because the $\langle 0\rangle$ and $\langle 1\rangle$ cover these.
3. $\alpha(n) := 1 - \alpha(n)$.
4. Go back to (2).

Outcomes for $\mathcal{R}_x$. As we show in the verification, there is only one final outcome, namely waiting at (2) forever; there is no infinite cycling through the plan's loop. In this case, $0.\alpha \neq \lim_{s\to\infty} x_s$.

Construction. We do a simple (relatively speaking) tree construction (see Lempp's notes [4] for instance). The requirements are ordered effectively with order type ω, and requirement R is assigned to a worker sitting on level/node R of a unary branching tree (which grows down, say).

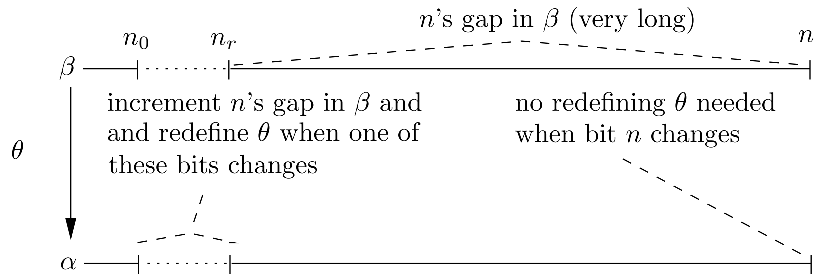

Figure 1. How bits change.

At each stage $t \in \mathbb{N}^+$ of the construction, workers $s < t$ act in order down the tree carrying out their plans from where they last left off at the previous stage up until time t, with the last/new worker at level $t-1$ beginning at step (1). Here time is measured by the number of stages in the simulation of the Turing machines involved. At each stage each worker has only one current outcome, namely waiting at step (2) (steps (3) and (4) do not count as using up time).

In this construction the workers do not really interfere with each other; there is no firing or rehiring of workers depending on different current outcomes. When higher priority workers (closer to the top/root of the tree) change α or β, lower priority workers (farther from the top/root of the tree) deal with the behavior easily by incrementing their gaps in β according to step (2).

Verification. Each worker on the final (only) schedule satisfies its $\mathcal{R}_x$ requirement.

To see this, fix a worker with plan $\mathcal{R}_x$ and bigbit n. When the worker goes from (2) to (3), bit n of α flips so that (the current approximation of) $0.\alpha$ changes/jumps by 2^{-n-1}. For the worker to reach (3) again (the current approximation of) x must jump by more than $2^{-n-2} = 2^{-n-1} - 2\epsilon$ to get back inside $0.\alpha$'s ϵ-ball. Actually, x might not jump by that much, because once x is outside of $0.\alpha$'s ϵ-ball, $0.\alpha$ might move toward x due to α's other bigbit flips. However, considering these flips, we get that x jumps by more than 2^{-n-3} (bigbit flips by bits smaller than n, move $0.\alpha$ tremendously so that x will certainly have to jump by more than 2^{-n-2}; bigbit flips by bits bigger than n move $0.\alpha$ by less than 2^{-n-3} (a bound obtained from a simple geometric series calculation; remember that bigbits are chosen extremely far apart from each other) so that x jumps by more than $2^{-n-3} = 2^{-n-2} - 2^{-n-3}$).

Now, to maintain the condition $\sum |x_s - x_{s+1}| \leq 1$, x can jump by $\geq 2^{-n-3}$ only $\leq 2^{n+3}$ times. Thus after 2^{n+3} passes from step (2) to (3) in the plan's loop, the worker must wait forever at (2).

Also, $\alpha = [\theta, 2]^\beta$ by construction.

Lastly, $0.\beta$ is a well-defined c.e. real. Each n_i flips $\leq 2^{n_i+3}$ times and each flip increments n's gap in β by 2^{-n}, but the gap is at least $\sum_{i \leq n} 2^{2n_i}$ long and is therefore big enough to absorb these additions without spilling carry bits into other gaps. ⊣

Proof of 4.2. We construct $\alpha \in {}^{\mathbb{N}}2$ as the characteristic function of a Δ^0_2 set such that $\alpha \not\leq_{\mathrm{rK}} \Omega$ via simple diagonalization. By Lemma 3.2, $0.\alpha$ will be a c.a. real.

Let $\#$ be a computable bijection from the set of all triples $\langle \varphi, i, c\rangle$, where φ is a c.p.f. from ${}^{<\mathbb{N}}2 \times \mathbb{N}$ to ${}^{<\mathbb{N}}2$ and $i < c$ are natural numbers. Also, let l be the function defined by $l(\varphi, c) = \max\{\#(\varphi, i, c) : i < c\} + 1$.

Now, using oracle $\emptyset'$ we define α as follows. If $\varphi(\Omega \restriction l(\varphi, c), i)\downarrow$ and is of length $l(\varphi, c)$, then let

$$\alpha(\#(\varphi, i, c)) = 1 - \varphi(\Omega \restriction l(\varphi, c), i)(\#(\varphi, i, c)).$$

Otherwise, let $\alpha(\#(\varphi, i, c)) = 0$.

For all pairs $\langle \varphi, c\rangle$, $\alpha \neq [\varphi, c]^{\Omega}$, because for all $i < c$, $\alpha \restriction l(\varphi, c) \neq \varphi(\Omega \restriction l(\varphi, c), i)$, as witnessed by bit $\#(\varphi, i, c)$. Thus $\alpha \not\leq_{\mathrm{rK}} \Omega$. ⊣

Let us end with one last question. We now know that $\mathbb{R}_{\text{d.c.e.}} \subset \mathbb{R}_{\Omega} \subset \mathbb{R}_{\text{c.a.}}$. Is $\mathbb{R}_{\text{d.c.e.}}$ or $\mathbb{R}_{\text{c.a.}}$ a randomness class? That is, does $\mathbb{R}_{\text{d.c.e.}}$ or $\mathbb{R}_{\text{c.a.}}$ equal $\mathbb{R}_y$ for *any* real number y?

By the proper inclusion of Theorem 4.1 and the technique in the proof of Theorem 4.2, it follows that, here again, the answer is negative.

Theorem 4.3. For all $y \in \mathbb{R}$, $\mathbb{R}_{\text{d.c.e.}} \neq \mathbb{R}_y$.

Theorem 4.4. For all $y \in \mathbb{R}$, $\mathbb{R}_{\text{c.a.}} \neq \mathbb{R}_y$.

Proof of 4.3. Assume (toward a contradiction) that for some $y \in \mathbb{R}$, $\mathbb{R}_{\text{d.c.e.}} = \mathbb{R}_y$. Since $\Omega \in \mathbb{R}_{\text{d.c.e.}} = \mathbb{R}_y \subseteq \mathbb{R}_{\Omega}$, $\Omega \leq_{\mathrm{rK}} y \leq_{\mathrm{rK}} \Omega$, so that $y \equiv_{\mathrm{rK}} \Omega$. Thus $\mathbb{R}_{\text{d.c.e.}} = \mathbb{R}_y = \mathbb{R}_{\Omega}$, a contradiction. ⊣

Proof of 4.4. Assume (toward a contradiction) that for some $y \in \mathbb{R}$, $\mathbb{R}_{\text{c.a.}} = \mathbb{R}_y$. Thus every c.a. real is $\leq_{\mathrm{rK}} y$. But carrying out the same construction as in the proof of Theorem 4.2 with y in place of Ω —note that in the proof no special properties of Ω, besides it being $\leq_{\mathrm{T}} \emptyset'$, were used— yields a c.a. real $\not\leq_{\mathrm{rK}} y$, a contradiction. ⊣

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Appendix. The anonymous referee has suggested some interesting alternative proofs of the results of section 4. Based on the definition of rK-redubility (in terms of conditional prefix-free complexity), they offer a different and valuable perspective.

Alternative proof of Theorem 4.1. Let Ω^{shift} be the shift of powers-of-two-position bits of Ω (as a binary sequence), that is, for $n \in \mathbb{N}$

$$\Omega^{\text{shift}}(n) := \begin{cases} \Omega(2n) & \text{if } n = 2^m \text{ for some } m \in \mathbb{N} \\ \Omega(n) & \text{otherwise.} \end{cases}$$

Notice that $\Omega^{\text{shift}} \in \mathbb{R}_\Omega$ since at each length n, a program using $\Omega \restriction n$ needs to guess only one bit to compute $\Omega^{\text{shift}} \restriction n$. However, $\Omega^{\text{shift}} \notin \mathbb{R}_{\text{d.c.e.}}$, so that $\mathbb{R}_{\text{d.c.e.}} \neq \mathbb{R}_\Omega$.

To see this, assume (toward a contradiction) that $\Omega^{\text{shift}} \in \mathbb{R}_{\text{d.c.e.}}$. From this assumption we will build a computable set C and a c.p.f. φ such that for all $m \in C$

$$\varphi(\Omega \restriction m) = \Omega(m).$$

This contradicts the fact that Ω is random (in the sense of c.e. martingales).

Since $\Omega^{\text{shift}} \in \mathbb{R}_{\text{d.c.e.}}$, there is a computable sequence of rationals $\langle \Omega^{\text{shift}} : s \in \mathbb{N}\rangle$ converging to Ω^{shift} such that $J := \sum_{s\in\mathbb{N}} |\Omega^{\text{shift}}_s - \Omega^{\text{shift}}_{s+1}| < \infty$. Let $\langle \Omega : s \in \mathbb{N}\rangle$ be an increasing sequence of rationals converging to Ω and for each $t \in \mathbb{N}$, let $J_t = \sum_{s=0}^{t-1} |\Omega^{\text{shift}}_s - \Omega^{\text{shift}}_{s+1}|$ with $J_0 = 0$. Notice that $\langle J : t \in \mathbb{N}\rangle$ is a computable sequence of rationals converging increasingly to J. Now, define two sequences of natural numbers $\langle s : n \in \mathbb{N}\rangle$ and $\langle t : n \in \mathbb{N}\rangle$ as follows.

$$\begin{aligned} s_n &:= \text{the least } s \text{ such that } \Omega_s \restriction 2^{n+1} = \Omega \restriction 2^{n+1} \text{ and} \\ &\quad \Omega^{\text{shift}}_s \restriction I = \Omega^{\text{shift}} \restriction I, \text{ where } I = [0, 2^{n+1}) \setminus \{2^n\}; \\ t_n &:= \text{the least } t \text{ such that } J_t \restriction (2^n + 2) = J \restriction (2^n + 2). \end{aligned}$$

Lastly, let $A = \{n \in \mathbb{N} : s_n \leq t_n\}$ and $B = \{n \in \mathbb{N} : n \geq 1 \wedge s_n > t_n\}$.

Notice that if $s_n \leq t_n$, then $\Omega \restriction 2^{n+1}$ can be computed from $J \restriction (2^n + 2)$, for given $J \restriction (2^n + 2)$ we can compute the least t ($= t_n$) such that $J_t \restriction (2^n + 2) = J \restriction (2^n + 2)$. Then, since $s_n \leq t_n$, $\Omega_t \restriction 2^{n+1} = \Omega \restriction 2^{n+1}$. Thus, by the characterization of rK-reducibility from Theorem 2.2, there is a constanst c_0 such that for all $n \in A$ $K(\Omega \restriction 2^{n+1} | J \restriction (2^n + 2)) \leq c_0$, implying that there are constants c_1, c_2, c_3 such that for all $n \in A$

$$\begin{aligned} K(\Omega \restriction 2^{n+1}) &\leq K(J \restriction (2^n + 2)) + c_1 \\ &\leq 2^n + 2 + 2\lg(2^n + 2) + c_2 \\ &\leq 2^n + 2n + c_3. \end{aligned}$$

Since Ω is random, this can happen for only finitely many n. So A is finite. Thus B is cofinite, hence computable.

Let C be the computable set $\{2^{n+1} : n \in B\}$ and let φ be the c.p.f. defined by

$$\varphi(\sigma) := \Omega^{\text{shift}}_s(2^n)$$

if s is the least number such that $\Omega_s \restriction 2^{n+1} = \sigma$ and $\Omega^{\text{shift}}_s \restriction I = \sigma^{\text{shift}} \restriction I$ (where the shift operation and I are as above), and let φ be undefined if there is no such s. Then for $2^{n+1} \in C$

$$\varphi(\Omega \restriction 2^{n+1}) = \Omega^{\text{shift}}_s(2^n) = \Omega^{\text{shift}}_{s_n}(2^n).$$

by definition of φ and s_n.

If $\Omega^{\text{shift}}_{s_n}(2^n) \neq \Omega^{\text{shift}}(2^n)$, then there is an $m > s_n$ such that $\Omega^{\text{shift}}_m \restriction 2^{n+1} = \Omega^{\text{shift}} \restriction 2^{n+1}$. Since $\Omega^{\text{shift}}_{s_n} \restriction 2^{n+1}$ and $\Omega^{\text{shift}}_m \restriction 2^{n+1} = \Omega^{\text{shift}} \restriction 2^{n+1}$ differ only on bit 2^n and $n \geq 1$, we have

$$J_m - J_{s_n} \geq |\Omega^{\text{shift}}_m - \Omega^{\text{shift}}_{s_n}| > 2^{-2^n-2}$$

so that $J_m \upharpoonright (2^n+2) \neq J_{s_n} \upharpoonright (2^n+2)$. This is a contradiction, since $J_{s_n} \upharpoonright (2^n+2) = J_m \upharpoonright (2^n+2) = J \upharpoonright (2^n+2)$ since $t_n < s_n$ for $n \in B$.

Thus for $2^{n+1} \in C$, $\varphi(\Omega \upharpoonright 2^{n+1}) = \Omega^{\text{shift}}_{s_n}(2^n) = \Omega^{\text{shift}}(2^n) = \Omega(2^{n+1})$, that is, for all $m \in C$

$$\varphi(\Omega \upharpoonright m) = \Omega(m),$$

a contradiction. ⊣

Alternative proof of Theorem 4.2. Let Ω^{even} and Ω^{odd} be the the even-position bits of and the odd-position bits of Ω (as a binary sequence), respectively, that is, for $n \in \mathbb{N}$

$$\Omega^{\text{even}}(n) := \Omega(2n) \qquad \text{and}$$
$$\Omega^{\text{odd}}(n) := \Omega(2n+1).$$

Notice that $\Omega^{\text{even}}, \Omega^{\text{odd}} \in \mathbb{R}_{\text{c.a.}}$ since $\Omega \in \mathbb{R}_{\text{c.a.}}$. However, both Ω^{even} and Ω^{odd} can not be in $\mathbb{R}_\Omega$, so that $\mathbb{R}_\Omega \neq \mathbb{R}_{\text{c.a.}}$.

To see this, assume (toward a contradiction) that $\Omega^{\text{even}}, \Omega^{\text{odd}} \in \mathbb{R}_\Omega$. Then, by the characterization of rK-reducibility from Theorem 2.2, there are constants c_0 and c_1 such that for all n $K(\Omega^{\text{even}} \upharpoonright n | \Omega \upharpoonright n) \leq c_0$ and $K(\Omega^{\text{odd}} \upharpoonright n | \Omega \upharpoonright n) \leq c_1$. Thus there is a c_2 such that for all n $K(\Omega \upharpoonright 2n | \Omega \upharpoonright n) \leq c_2$ (by combining the two underlying algorithms), implying that there are c_3, c_4 such that for all n

$$K(\Omega \upharpoonright 2n) \leq K(\Omega \upharpoonright n) + c_3 \leq n + 2\lg n + c_4.$$

This is a contradiction, since Ω is random. ⊣

Alternative proof of Theorem 4.4. Actually, we show that for all $y \in \mathbb{R}$, $\mathbb{R}_{\text{c.a.}} \not\subseteq \mathbb{R}_y$. Assume (toward a contradiction) that for some $y \in \mathbb{R}$, $\mathbb{R}_{\text{c.a.}} \subseteq \mathbb{R}_y$. Then $\Omega^{\text{even}}, \Omega^{\text{odd}} \in \mathbb{R}_{\text{c.a.}} \subseteq \mathbb{R}_y$. However, this is a contradiction since, by the same proof as above with $y \upharpoonright n$ in place of $\Omega \upharpoonright n$, both Ω^{even} and Ω^{odd} can not be in $\mathbb{R}_y$. ⊣

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