A MINIMAL RK-DEGREE

ALEXANDER RAICHEV AND FRANK STEPHAN

Abstract. We construct a minimal rK-degree, continuum many, in fact. We also show that every minimal sequence, that is, a sequence with minimal rK-degree, must have very low descriptional complexity, that every minimal sequence is rK-reducible to a random sequence, and that there is a random sequence with no minimal sequence rK-reducible to it.

1. Introduction

This article continues the study of relative randomness via rK-reducibility initiated in [DHL04] and pursued in [Rai05].

One of the most popular definitions of absolute algorithmic randomness states that an infinite binary sequence R is random if it is incompressible, that is, if

$$\exists d\, \forall n \,.\, K(R \restriction n) \geq n - d,$$

where $K(\sigma)$ is the prefix-free descriptional complexity of the string σ. Under this same paradigm of incompressibility, one can define relative algorithmic randomness as follows. An infinite binary sequence A is less random than an infinite binary sequence B if A is completely compressible given B, that is, if

$$\exists d\, \forall n \,.\, K(A \restriction n | B \restriction n) < d,$$

where $K(\sigma|\tau)$ is the conditional prefix-free descriptional complexity of σ given τ. In this case, we write $A \leq_{\mathrm{rK}} B$ for short and say "A is rK-reducible to B".[1]

The $\leq_{\mathrm{rK}}$ relation, which is fairly easily seen to be reflexive and transitive, enjoys the following properties, all of which we will use throughout.

Theorem 1.1 ([DHL04]). For infinite binary sequences A and B, $A \leq_{\mathrm{rK}} B$ is equivalent to both of

- $\exists d\, \forall n \,.\, C(A \restriction n | B \restriction n) < d$
- there exists a computable partial function φ such that $\exists d\, \forall n\, \exists i < d \,.\, \varphi(i, B \restriction n) = A \restriction n$

and implies all three of

- $\exists d\, \forall n \,.\, K(A \restriction n) \leq K(B \restriction n) + d$
- $\exists d\, \forall n \,.\, C(A \restriction n) \leq C(B \restriction n) + d$
- $A \leq_{\mathrm{T}} B$

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[1] The 'rK' stands for 'relative Kolmogorov' complexity, another name for conditional prefix-free descriptional complexity.

Notice (from the second bullet, say) that a computable sequence is rK-reducible to any given sequence. Also, from the fifth bullet, any sequence rK-reducible to a computable sequence is itself computable. So the computable sequences are those of least relative randomness, as they should be.

In what follows we answer a basic question: is there a sequence of minimal relative randomness, that is, a sequence with only the computable sequences strictly less random ($<_{\mathrm{rK}}$) than it? Indeed, as our title indicates, there is. In fact, there are continuum many. These are our main results, which we prove in Section 2, and we follow them with three notes on such minimal sequences in Section 3.

Before beginning, let us set some notation and conventions. $\mathbb{N}$ will denote the set of natural numbers $\{0, 1, 2, \ldots\}$, ${}^{<\mathbb{N}}2$ the set of binary strings, and ${}^{\mathbb{N}}2$ the set of infinite binary sequences. 'String' and 'sequence' without further qualification will mean 'binary string' and 'infinite binary sequence', respectively. For strings σ and τ, $|\sigma|$ will denote the length of σ, and $\sigma\tau$ or, when that might cause confusion, $\sigma^\frown\tau$ the concatenation of σ and τ. Also $\sigma \subseteq \tau$ and $\sigma \subset \tau$ will mean σ is a initial segment of τ and σ is a proper initial segment of τ, respectively. For a sequence A and a positive natural n, $A \upharpoonright n$ will denote the length n initial segment of A, that is, the string $\langle A(0), A(1), \ldots, A(n-1)\rangle$. Trees are subsets of ${}^{<\mathbb{N}}2$ closed under initial segments. A path of a tree T is a sequence, every initial segment of which lies on/is a member of T. The set of all paths of T will be denoted by $[T]$. A Π^0_1 tree is a tree whose complement is computably enumerable, and a Π^0_1 class is the set of all paths through such a tree. Lastly, our notation for computability-theoretic notions follows that of [Soa87] and [Odi89].

2. The main results

Theorem 2.1. There is a minimal rK-degree.

Proof. We construct a special binary tree, suitable paths of which will have minimal rK-degree. Roughly speaking, we make the set of splitting nodes of our tree very sparse so that any incomputable path of hyperimmune-free Turing degree can be recovered in two guesses from its image under an rK-reduction. More precisely, we build a Π^0_1 tree T such that

(1) T has no computable paths;
(2) for every computable function $\Phi : \mathbb{N} \to \mathbb{N}$ (thought of as a functional) and for every path X of T there is a string $\star \subset X$ such that either
  (a) for every path Y of T extending $\star$, $\Phi^Y = \Phi^X$, or
  (b) for all pairs of distinct paths Y, Z of T extending $\star$, Φ^Y and Φ^Z are incompatible;
(3) the set S of splitting nodes of T is very sparse, to wit, for all computable functions $g : \mathbb{N} \to \mathbb{N}$ we have

$$\forall^\infty \sigma \in S\ \forall \tau \in S\ .\ \sigma \subset \tau \to g(|\sigma|) < |\tau|.$$

Constructing T. We build T in stages, beginning with the full binary tree and pruning it computably. To describe this pruning we use moving markers in the style of [Ste01]. For notational niceness stage subscripts are suppressed whenever possible.

Let $\{m_\sigma : \sigma \in {}^{<\mathbb{N}}2\} \subseteq {}^{<\mathbb{N}}2$ denote the set of markers of T. These are/lie on the splitting nodes of T. At stage zero, $T = {}^{<\mathbb{N}}2$ and each $m_\sigma = \sigma$. At later stages when necessary T is pruned via the CUT procedure. For $\sigma \subset \tau$ $\textsc{Cut}(m_\sigma, m_\tau)$ cuts off all paths of T that extend m_σ but not m_τ and then updates the positions of all the markers, preserving their order, as follows: m_σ moves to m_τ, each $m_{\sigma\epsilon}$ moves to $m_{\tau\epsilon}$, and all other markers stay put. Since Cut is the only action ever taken, T will be a perfect tree without leaves at every stage.

At stage $s > 0$ the construction runs as follows, where each check is performed only when the markers involved have indices of length $\leq s$; also, the computations involved are only up to stage s.

- If there exist σ, $i < 2$, and $e \leq |\sigma|$ such that for all $x \leq |\sigma|$, $\Phi_e(x) = m_{\sigma i}(x)$, then $\textsc{Cut}(m_\sigma, m_{\sigma(1-i)})$.
- If there exist σ, δ, ϵ, and $e \leq |\sigma|$ such that $\Phi_e^{m_{\sigma 0}}$ and $\Phi_e^{m_{\sigma 1}}$ are compatible for all arguments $\leq |\sigma|$, but $\Phi_e^{m_{\sigma 0\delta}}$ and $\Phi_e^{m_{\sigma 1\epsilon}}$ are incompatible at some argument $\leq |\sigma|$, then $\textsc{Cut}(m_{\sigma 0}, m_{\sigma 0\delta})$ and $\textsc{Cut}(m_{\sigma 1}, m_{\sigma 1\epsilon})$.
- If there exist σ, τ, υ, and $e \leq |\sigma|$ such that $\sigma \subset \tau \subset \upsilon$ and $|m_\tau| \leq \Phi_e(|m_\sigma|) < |m_\upsilon|$, then $\textsc{Cut}(m_\tau, m_\upsilon)$.

It is not difficult to check that each marker eventually settles and that, in the end/limit, T satisfies properties (1)-(3).

A suitable path of T. Let A be a path of T of hyperimmune-free Turing degree.[2] Such a path exists by the Hyperimmune-free Basis Theorem ([JS72]) since $[T]$ is a nonempty Π^0_1 class. We show that A has minimal rK-degree. By (1) A is incomputable. Let $B \leq_{\mathrm{rK}} A$ be a incomputable set. We need to show that $A \leq_{\mathrm{rK}} B$. To this end, observe that $B \leq_{\mathrm{T}} A$, and, in fact, $B \leq_{tt} A$ since A has hyperimmune-free Turing degree (see [Odi89, page 589]). Let Φ be a computable functional (total on all oracles) that witnesses the truth-table reduction.

We come now to the heart of the argument: building an rK-reduction from B to A. Let $\star$ be the magic string of (2) for Φ and A. Given $B \upharpoonright n$ for n sufficiently large, run through the computable approximation (that thins) to T until a stage t is reached such that T_t (the stage t approximation of T) has at most two extensions of $\star$ of length n with extensions in T_t that map to $B \upharpoonright n$ under Φ. The key here is that such a stage is guaranteed to exist by Lemma 2.2 below. To find these extensions and extensions computably from $B \upharpoonright n$ we use the fact that Φ is total on all oracles and has a computable use function. Output the (at most) two strings of length n found; one will be $A \upharpoonright n$. Except for finitely many short lengths, this procedure describes an rK-reduction from B to A. Extending it to all lengths gives the final reduction. □

Lemma 2.2. Let $\star$ be the magic string of (2) for A. For almost all lengths n and almost all stages t, T_t has at most two extensions of $\star$ of length n with extensions in T_t that map to $B \upharpoonright n$ under Φ.

Proof. Let φ be the computable use function for the tt-reduction Φ. Let f be the function defined for $m \geq |\star|$ by $f(m)$ equals the first stage s such that for all strings $\nu \supset A \upharpoonright m\widehat{\ }(1 - A(m))$ of length $\varphi(s)$ on T_s, there exists $x \leq s$ such that $\Phi^\nu(x)\downarrow \neq \Phi^A(x)$. (Notice that all ν extend $\star$.) For $m < |\star|$, define $f(m)$ to be 0, say. It is unimportant. Note that f is total, for if not, then for all s there exists a string $\nu_s \supset A \upharpoonright m\widehat{\ }(1 - A(m))$ of length $\varphi(s)$ on T_s such that for all $x \leq s$, $\Phi^{\nu_s}(x) = \Phi^A(x)$. (Remember that Φ is total on all oracles.) Then the sequence Y defined by $Y(n) = \liminf_s \nu_s(n)$ is a path of T different from A such that $\Phi^Y = \Phi^A$. But this is a contradiction, because (2b) holds for $X = A$ since B is incomputable.

[2]That is, for every total function $f \leq_{\mathrm{T}} A$, there exists a computable function g such that for all x, $g(x) \geq f(x)$. Put more concisely, every total function computable from A has a computable majorant.

Also, f is A-computable by definition. Thus, since A has hyperimmune-free Turing degree, there is a computable function g majorizing f.

Now, fix n bigger than the length of $\star$, the length that (3) takes effect for g, and the length of the first splitting node of A on T. Let τ be the last splitting node of T on $A \upharpoonright n$, and let $\sigma \subset \tau$ be any other splitting node of T extending $\star$. Then by (3) we have that

$$s := f(|\sigma|) \leq g(|\sigma|) < |\tau| \leq n.$$

So by stage s every string $\nu \in T_s$ extending $A \upharpoonright |\sigma| \widehat{\ }(1 - A(|\sigma|) = \sigma \widehat{\ }(1 - A(|\sigma|)$ will have some number $x \leq s < n$ such that $\Phi^{\nu}(x)\downarrow \neq \Phi^{A}(x) = B(x)$, so that ν cannot map to $B \upharpoonright n$ under Φ. Since σ was an arbitrary splitting node of T below the last splitting node of $A \upharpoonright n$, we see that only the strings extending the last splitting node of $A \upharpoonright n$ can map to $B \upharpoonright n$ under Φ. Similarly, by considering $s' := f(|\tau|) \leq n$, any splitting node of T of length n extending τ can not have extensions in T'_s mapping to $B \upharpoonright n$ under Φ. So the result holds. □

In fact, by a generalized hyperimmune-free basis theorem below, the tree of the proof of Theorem 2.1 has continuum many paths of hyperimmune-free Turing degree. Thus, since every rK-degree is countable, there are continuum many minimal rK-degrees.

Theorem 2.3. Every nonempty Π^0_1 class with no computable members has $2^{\aleph_0}$ paths of hyperimmune-free Turing degree.

Proof. By basic facts from the theory of Π^0_1 classes, we can assume without loss of generality that our Π^0_1 class is the set of paths through a tree T_0 that is infinite, computable, and has no computable paths. We modify slightly the proof of the Hyperimmune-free Basis Theorem in [JS72] by way of an extra parameter sequence X. For each sequence X we construct (computably in $X \oplus \emptyset''$) computable subtrees $S_1 \supset T_1 \supseteq S_2 \supset T_2 \supseteq \cdots$ of T_0 such that their only common path Y has hyperimmune-free Turing degree. We then show that the map $X \mapsto Y$ is one-to-one.

To this end, fix X and, starting from T_0, let S_e and T_e be defined recursively as follows. Let $U_{e,x}$ be the computable tree $\{\tau : \Phi^{\tau}_{e,|\tau|}(x)\uparrow\}$.

(1) If for all x, $T_e \cap U_{e,x}$ is finite, then $S_e := T_e$. Otherwise, choose x least such that $U_{e,x}$ is infinite and $S_e := T_e \cap U_{e,x}$.
(2) Since S_e is an infinite tree with no computable paths, it has at least two paths. Let σ be the length-lexicographic least node of S_e such that $\sigma 0$ and $\sigma 1$ have paths in S_e through them.
(3) $T_{e+1} := \{\tau \in S_e\}\tau \subseteq \sigma \widehat{\ } X(e) \vee \tau \supset \sigma \widehat{\ } X(e)$.

By induction each $[T_e]$ and $[S_e]$ is nonempty, so that $\bigcap_e [T_e] \cap [S_e]$ is nonempty, being the intersection of a decreasing sequence of closed nonempty sets in the compact space ${}^{\mathbb{N}}2$. Choose (the unique) sequence $Y \in \bigcap_e [T_e] \cap [S_e]$. It will have hyperimmune-free Turing degree, for fix a natural e and consider the function Φ^Y_e. If for every x, $T_e \cap U_{e,x}$ is finite, then the following function is total, computable, and majorizes Φ^Y_e.

$$g(x) = \max\{\Phi^{\tau}_{e,|\tau|}(x)\}\tau \in T_e \wedge |\tau| = l_x,$$

where l_x is least such that $\Phi^{\tau}_{e,|\tau|}(x)$ is defined for each $\tau \in T_e$ of length l_x. If there exists some x such that $T_e \cap U_{e,x}$ is infinite, then $\Phi^{\tau}_{e,|\tau|}(x)$ is undefined for infinitely many $\tau \in T_e$, and S_e

is the set of all these τ. Since all prefixes of Y are in S_e, this means $\Phi_e^Y(x)$ is undefined, so that Φ_e^Y is not total.

Also, the map $X \mapsto Y$ is one-to-one, for if two sequences X_1 and X_2 differ, and e is the first place at which this happens, then the corresponding trees $S_e(X_1)$ and $S_e(X_2)$ are the same, but the intersection of $T_{e+1}(X_1)$ and $T_{e+1}(X_2)$ is finite since one contains the nodes above $\sigma 0$ and the other the ones above $\sigma 1$. Thus $Y(X_1)(|\sigma|) \neq Y(X_2)(|\sigma|)$. □

3. Three notes

From now on let us call a sequence with minimal rK-degree a 'minimal sequence'. As one might expect, minimal sequences have low initial segment complexity. Indeed, so low that they are close to being computable in the sense of Chaitin's characterization (see [Cha76]): a sequence X is computable iff $\exists d\, \forall n \,.\, C(X \restriction n) \leq C(n) + d$.

Proposition 3.1. If A is a minimal sequence, then for any computable unbounded increasing function $g : \mathbb{N} \to \mathbb{N}$,

$$\exists d\, \forall n \,.\, C(A \restriction n) \leq C(n) + g(n) + d \quad \text{and}$$
$$\exists d\, \forall n \,.\, K(A \restriction n) \leq K(n) + g(n) + d.$$

In particular, A cannot be random.

We prove this with dilutions.

Definition 3.2. For $X \in {}^{\mathbb{N}}2$ and $f : \mathbb{N} \to \mathbb{N}$ strictly increasing, the *f-dilution of X* is the sequence defined by

$$X_f(n) = \begin{cases} X(m) & \text{if } n = f(m) \text{ for some (unique) } m \\ 0 & \text{else.} \end{cases}$$

Notice that for any sequence X and any strictly increasing computable function f, $X_f \leq_{\mathrm{rK}} X$ and $X_f \equiv_{\mathrm{T}} X$.

Proof of Proposition 3.1. Fix A and g as in the hypothesis. The idea is that since A is a minimal sequence, it is rK-reducible to every one of its computable dilutions. Picking a dilution appropriate to g will give the desired complexity bound.

We prove the bound for K. The argument for C is identical. Define the function $f : \mathbb{N} \to \mathbb{N}$ recursively by

$$f(0) = 0;$$
$$f(x) = \text{the least } n \text{ such that } n > f(x-1) \text{ and } g(n) \geq 4x.$$

Since g is unbounded and increasing, f is well-defined. Also, by construction f is computable, total, and strictly increasing. Furthermore, for any given n, if x is the greatest number such that $f(x) \leq n$, then $g(n) \geq 4x$.

Since A is minimal, $A \leq_{\mathrm{rK}} A_f$ via some $[\varphi, e]$. Now fix n and choose x greatest such that $f(x) \leq n$. Observe that inserting zeros into $A \restriction x$ in the appropriate computable places produces $A_f \restriction n$. So to describe $A \restriction n$, besides a few computable partial functions given ahead of time, one only needs the correct $i < e$ such that $\varphi(i, A_f \restriction n) = A \restriction n$, the value n and $A \restriction x$. This information can be coded, up to a uniform constant, by a string of length

$K(n)+2K(A\upharpoonright x)$. The factor of 2 comes from concatenating strings in a prefix-free way. So, up to a uniform additive constant, for all n,

$$K(A\upharpoonright n) \leq K(n) + 2K(A\upharpoonright x) \leq K(n) + 4x \leq K(n) + g(n),$$

as desired. Now fixing g as, say, $g(n) = \lfloor \lg(n+1) \rfloor$, we see that A cannot be random. □

Using dilutions again, we also get the following.

Proposition 3.3. Every minimal sequence is rK-reducible to a random sequence.

Proof. Fix a minimal sequence A, and choose a random sequence $R \geq_{\mathrm{wtt}} A$ with use majorized by $f(n) = 2n$. This is possible since every sequence has such a random ([Kuč85],[Gác86]; see also [MM04] for a more recent proof using martingales). Then $R \geq_{\mathrm{rK}} A_f \geq_{\mathrm{rK}} A$, by the minimality of A, as desired. □

Do all sequences have randoms rK-above them? That question is still open and seemingly difficult.

We end with one last note, a contrast to Proposition 3.3.

Proposition 3.4. There is a random sequence with no minimal sequence rK-reducible to it.

Proof. Let R be a random sequence of hyperimmune-free Turing degree. Such a sequence exists by the Hyperimmune-free Basis Theorem applied to the complement of any member of a universal Martin-Löf test. Then R has no minimal sequence reducible to it.

To see this, assume (toward a contradiction) there is some minimal sequence A such that $A \leq_{\mathrm{rK}} R$. Since R has hyperimmune-free Turing degree, so does A and $A \leq_{\mathrm{tt}} R$. Since A is incomputable and truth-table reducible to a random sequence, A is Turing equivalent to some random sequence S (see [Dem88]). Since A has hyperimmune-free Turing degree, $S \leq_{\mathrm{tt}} A$ via some computable partial function with computable use function f. Thus, disregarding floor functions and uniform constants for ease of reading, we have that for all n

$$\begin{aligned} n &\leq K(S\upharpoonright n) && \text{(since } S \text{ is random)} \\ &\leq 2K(A\upharpoonright f(n)) && \text{(using the tt-reduction)} \\ &\leq 2K(f(n)) + 2\lg n && \text{(by Proposition 3.1)} \\ &\leq 2K(n) + 2\lg n && \text{(since } f \text{ is computable)} \\ &\leq 4\lg n, \end{aligned}$$

a contradiction. □

Remark 3.5. Do maximal rK-degrees exist? That basic question is still open.

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Department of Computer Science, University of Auckland, Private Bag 92019, Auckland, New Zealand

E-mail address: raichev@cs.auckland.ac.nz

Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543

E-mail address: fstephan@comp.nus.edu.sg