New software for computing asymptotics of multivariate generating functions

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Abstract

I introduce `amgf`, a new Sage software package for computing asymptotics of multivariate generating functions. It implements recent algorithms developed by Mark C. Wilson and me. The current version of `amgf` is under peer review for incorporation into Sage and is available from my website at `www.cs.auckland.ac.nz/~raichev/research.html`.

1 Introduction

Let $F(x) = \sum_{\nu \in \mathbb{N}^d} F_\nu x_1^{\nu_1} \cdots x_d^{\nu_d}$ be a multivariate power series with complex coefficients that converges in a neighborhood of the origin. Assume $F = G/H$ for some functions G and H holomorphic in a neighborhood of the origin. For example, F could be the combinatorial generating function $(1 - x_1 - x_2 - x_1 x_2)^{-1}$ whose power series coefficients F_{ν_1,ν_2} (the Delannoy numbers) count the number of lattice paths from $(0,0)$ to (ν_1, ν_2) with allowable steps $(1,0)$, $(0,1)$ and $(1,1)$.

Oftentimes one would like an asymptotic expansion for the coefficients F_ν as $\nu \to \infty$ along various paths. Perhaps one is curious about the growth rate of the coefficients, in which case an asymptotic expansion would be handy. The best general method for deriving such an expansion is singularity analysis, a powerful technique from analytic combinatorics which explains how the type and location of the singular points $\mathcal{V} := \{x \in \mathbb{C}^d : H(x) = 0\}$ of F determine the asymptotics of F_ν. Singularity analysis in the univariate setting ($d = 1$) is well-developed; see the book [FS09] for an encyclopedic introduction. Singularity analysis in the multivariate setting ($d > 1$) is just beginning to bloom; see the survey article [PW08] for an introduction. My focus here is on the multivariate theory.

When points of the analytic variety $\mathcal{V}$ of 'smooth' or 'multiple' type predominate, the general form of the asymptotic expansion of the ray coefficients $F_{n\alpha}$ as $n \to \infty$ and α remains in a permissible subset of $\mathbb{R}^d_+$ was derived in [PW02, PW04]. More recently, algorithms to explicitly calculate the coefficients of this general form have been developed in [RW08, RW]. Now these algorithms have been implemented in `amgf` for the case of polynomial H so that a user with basic knowledge of multivariate singularity analysis can compute arbitrary terms of the asymptotic expansion of $F_{n\alpha}$.

The `amgf` package is designed for the Sage mathematics software system [S+11], a viable free open-source alternative to Magma, Maple, Mathematica and Matlab*. The current version of `amgf` is under peer review for incorporation into Sage and is available from my website at `www.cs.auckland.ac.nz/~raichev/research.html`.

2 Example

Let us walk through an example of computing an asymptotic expansion with `amgf`. The software is designed to help with most of the computations involved, some of which are infeasible by hand alone. That said, one still needs to know basic singularity analysis to effectively use `amgf`.

Consider the trivariate rational function $F : \mathbb{C}^3 \to \mathbb{C}$ defined by

$$F(x, y, z) = \frac{1}{(1 - x(1 + y))(1 - zx^2(1 + 2y))}.$$

It is holomorphic around the origin, and its power series coefficients F_ν, which are all nonnegative, arise in computing the diameters of random Cayley graphs of groups [PW08, Example 4.10].

```
sage: R.<x,y,z>= PolynomialRing(QQ)
sage: G= 1
sage: H= (1-x*(1+y))*(1-z*x^2*(1+2*y))
sage: F= QuasiRationalExpression(G,H)
```

Let us compute the first three terms of the asymptotic expansion of $F_{n\alpha}$ as $n \to \infty$ for a specific value of α. Again, singularity analysis tells us that these asymptotics are determined by the type and location of the singular points $\mathcal{V} = \{(x, y, z) \in \mathbb{C}^3 : H(x, y, z) = 0\}$ of F. In this case, not all points of $\mathcal{V}$ are smooth. Some are singular (in the algebraic geometry sense), and the set of these is $\mathcal{V}' = \{(x, y, z) \in \mathbb{C}^3 : H(x, y, z) = \nabla H(x, y, z) = 0\} = \{(1/(a + 1), a, (a + 1)^2/(2a + 1)) : a \in \mathbb{C} \setminus \{-1\}\}$.

```
sage: I= F.singular_points(); I
Ideal (x*y + x - 1, y^2 - 2*y*z + 2*y - z + 1, x*z + y - 2*z + 1) of
Multivariate Polynomial Ring in x, y, z over Rational Field
sage: solve(I.gens(),[SR(x),SR(z)],solution_dict=true)
[{z: (y^2 + 2*y + 1)/(2*y + 1), x: 1/(y + 1)}]
```

This set consists entirely of convenient multiple points of order $r = 2$ [RW, Definition 2.2].

```
sage: a= var('a')
sage: sp= {z: (a^2 + 2*a + 1)/(2*a + 1), y: a, x: 1/(a + 1)}
sage: F.is_cmp(sp)
[({y: a, z: (a^2 + 2*a + 1)/(2*a + 1), x: 1/(a + 1)}, True, 'all good')]
```

The points of $\mathcal{V}'$ nearest the origin in terms of polydiscs are $(1/(a + 1), a, (a + 1)^2/(2a + 1))$ for $a > 0$. These points determine asymptotics for all directions α in their critical cone [RW, Definition 2.3], that is, for all directions in the conical hull of the vectors $\gamma_1 = (1, a/(a + 1), 0)$ and $\gamma_2 = (1, a/(2a + 1), 1/2)$.

*The `amgf` Sage package differs from and complements the `Mgfun` Maple package, which, as I understand it, provides tools for manipulating multivariate generating functions but does not compute multivariate asymptotics.

```
sage: F.critical_cone(sp)
[((a + 1)/a, 1, 0), ((2*a + 1)/a, 1, 1/2*(2*a + 1)/a)]
```

For instance, $c = (1/2, 1, 4/3)$ controls asymptotics for all α in the conical hull of the vectors $\gamma_1(c) = (2, 1, 0)$ and $\gamma_2(c) = (3, 1, 3/2)$. For instance, $\alpha = (8, 3, 3)$ is in this critical cone. By [RW, Theorem 3.5], which is called by `asy= F.asymptotics(c,alpha,3)`, we get

$$F_{n\alpha} = 108^n \left[\frac{3}{\sqrt{21\pi}} n^{-1/2} - \frac{1231}{8232\sqrt{21\pi}} n^{-3/2} + \frac{329047}{58084992\sqrt{21\pi}} n^{-5/2} + O(n^{-7/2}) \right]$$

as $n \to \infty$.

The command `F.relative_error(asy[0],alpha,[1,2,4,8],asy[1])` outputs the following table of relative errors between the numerical values of $F_{n\alpha}$ and its three term approximation $S_3(n)$ above for small values of n.

n	$F_{n\alpha}/c^{-n\alpha}$	$S_3(n)/c^{-n\alpha}$	relative error
1	0.3518518519	0.3516356189	0.0006145569473
2	0.2548010974	0.2547831876	0.00007028933620
4	0.1823964231	0.1823948602	0.000008568698736
8	0.1297748629	0.1297747255	0.000001058756657

References

[FS09] Phillipe Flajolet and Robert Sedgewick, *Analytic combinatorics*, Cambridge University Press, 2009.

[PW02] Robin Pemantle and Mark C. Wilson, *Asymptotics of multivariate sequences. I. Smooth points of the singular variety*, J. Combin. Theory Ser. A **97** (2002), no. 1, 129–161. MR MR1879131 (2003a:05015)

[PW04] ______, *Asymptotics of multivariate sequences. II. Multiple points of the singular variety*, Combin. Probab. Comput. **13** (2004), no. 4-5, 735–761. MR MR2095981 (2005i:05008)

[PW08] ______, *Twenty combinatorial examples of asymptotics derived from multivariate generating functions*, SIAM Rev. **50** (2008), no. 2, 199–272. MR MR2403050

[RW] Alexander Raichev and Mark C. Wilson, *Asymptotics of coefficients of multivariate generating functions: improvements for multiple points*, To appear.

[RW08] ______, *Asymptotics of coefficients of multivariate generating functions: improvements for smooth points*, Electron. J. Combin. **15** (2008), no. 1, Research Paper 89, 17. MR MR2426152

[S+11] W. A. Stein et al., *Sage Mathematics Software (Version 4.6.2)*, The Sage Development Team, 2011, `http://www.sagemath.org`.